EXPERIMENTS ON THE TWO-ENVELOPE PROBLEM

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The two-envelope paradox is a vexing probability problem. Its conclusion is so absurd that we are forced to question the validity of our mathematical approach.

A student in the residential version of *Paradox and Infinity* suggested running a simulation to verify the conclusion of the two-envelope paradox. That was also my thinking. Having worked with simulations before, I knew that they were quick and easy to program. I decided to run a computer simulation of the two-envelope problem in order to reassure myself that the conclusion was wrong (so probability theory is out of danger) and, more importantly, to identify where the calculations went awry.

Turns out that the calculations are pretty much all correct. Only the very last logical leap is wrong. In this paper, I intend to shed light on the two-envelope paradox. I will focus on Broome's variation of the problem and use the results of my simulation to identify which results hold and which do not.

The Two-Envelope Problem

Let's start by restating the two-envelope problem so we are all on the same page. I will also use this occasion to introduce the notation I will be using in this paper.

A curious man offers you to play a curious game. He chooses a natural number randomly (we will come back to this random choice in a moment). He puts that amount of money in one envelope, which we will call E_1. He then puts double that amount in another envelope, which we will call E_2.

The two envelopes are indistinguishable to you. You do not know which one is E_1 (and therefore contains the smaller amount) and which one is E_2 (and therefore contains the bigger amount). The curious man asks you to choose one of the two envelopes. The envelope you pick will be *your* envelope, and the envelope you leave out will be *his* envelope or *the other* envelope.

After you have made your choice, the curious man invites you to look at the contents of your envelope. You take a peek. You see that your envelope contains n dollars.*

Now comes the tricky part. The curious man offers you to either keep your envelope, or to switch and take his envelope. Should you keep or switch?

To answer this question, we need to specify how the curious man chooses his natural number. In the original problem, we assume that the distribution is uniform, i.e. each natural number has the same probability of getting picked.

The solution then goes like this: after seeing that your envelope contains n dollars, you know that the other envelope contains either $n/2$ dollars or $2n$ dollars. Given the uniform distribution, each of those two possibilities is equally likely. Your expected value if you keep your envelope is $EV[\text{keep}] = n$. Your expected value if you switch is

$$EV[\text{switch}] = \frac{n}{2}\frac{1}{2} + 2n\frac{1}{2} = n\frac{5}{4}.$$

*This formulation of the problem, where you get to know the contents of your envelope, is slightly easier to work with than the original formulation, where you contemplate but do not know the contents of your envelope. The calculations and the conclusion remain the same.

To maximize your expected value, you should switch. No matter what n you saw in your envelope, *you should always switch.* Therefore, you don't even need to look at the contents of your envelope to know that you should switch. That conclusion goes straight against common sense.

Worse still, if you looked at the contents of the other envelope but not yours, a similar reasoning would lead us to say you should always keep.

Broome's Variation

Simulating a uniform distribution on natural numbers is, I dare say, impossible. I thus focused on Broome's variation of the two-envelope problem. The setting is the same, except for the way the curious man chooses his natural number.

The curious man tosses a die. If he gets 1 or 2, he stops. If he gets 3, 4, 5 or 6, he tosses the die again. Let's say he stops after k tosses. The curious man then puts 2^{k-1} dollars in envelope E_1 and 2^k dollars in envelope E_2.

Once again, you get to pick an envelope and look inside. You see that it contains $n = 2^m$ dollars. Now you have to decide whether you want to keep your envelope or switch. To maximize your expected value, what should you do?

Let's get to work and calculate the conditional expectations. If $n = 1$, then you know for certain that the other envelope contains 2 dollars and you should definitely switch.

For $n > 1$, seeing $n = 2^m$ dollars in your envelope means that either $E_1 = 2^m$ or $E_2 = 2^m$. Knowing that you picked your envelope randomly and knowing how the curious man filled the envelopes, we can work out that

$$P[\text{other envelope} = 2^{m+1}] = P[\text{your envelope is } E_1 \text{ and } E_1 = 2^m] = \frac{2^{m-1}}{3^{m+1}}$$
$$P[\text{other envelope} = 2^{m-1}] = P[\text{your envelope is } E_2 \text{ and } E_2 = 2^m] = \frac{2^{m-2}}{3^m}.$$

Finally, we can calculate your expected value if you switch, given that your envelope contains $n = 2^m$ dollars.

$$EV[\text{switch}] = \frac{2^{m+1}\,P[\text{other envelope} = 2^{m+1}] + 2^{m-1}\,P[\text{other envelope} = 2^{m-1}]}{P[\text{other envelope} = 2^{m+1}] + P[\text{other envelope} = 2^{m-1}]}$$
$$= 2^{m+1}\,\frac{2}{5} + 2^{m-1}\,\frac{3}{5} = 2^m\,\frac{11}{10}$$

Once again, we reach the perplexing conclusion that you should always switch, no matter what amount $n = 2^m$ you saw in your envelope.

Simulation

I ran a simulation of Broome's variation of the two-envelope problem on my computer. The results were both disappointing and enlightening.

In short, it confirmed that once you know the contents of your envelope, you should always switch. Switching actually does increase your expected value.[†]

It is worth noting at this point that there is no paradox without conditioning on the contents of your envelope. The unconditional expectations are easily shown to be $EV[\text{keep}] = EV[\text{switch}] = \infty$.

[†]That was a surprise to me. I was expecting the simulation to show that you should be indifferent. I explored this result by changing the parameters of my simulation and I finally understood why it made sense, but that's a discussion for another day.

This unconditional indifference is borne out in my simulation. Let's use AV to denote the average value from my simulation, i.e. the empirical version of EV. In my simulation, when I stop at 500 runs, I obtain $AV[\text{keep}] \approx 105$ and $AV[\text{switch}] \approx 103$. Roughly speaking, the more runs I simulate, the higher the average values become (they do not converge), but neither $AV[\text{keep}]$ nor $AV[\text{switch}]$ dominates.

Moreover, there is no paradox if we condition on the number of die tosses, or on the contents of E_1 or E_2. That is also confirmed by my simulation results.

What about the expected value of keep and switch, given the contents of your envelope? For the most part, my simulation confirms the analytical results. For instance, if I look only at the simulation runs where your envelope ends up containing 2 dollars, I obtain $AV[\text{keep}] = 2$ and $AV[\text{switch}] \approx 2.2$. In runs where your envelope contains 4 dollars, my simulation yields $AV[\text{keep}] = 4$ and $AV[\text{switch}] \approx 4.4$. And so on.

Wait a second! How can the unconditional averages be equal if $AV[\text{switch}]$ is greater than $AV[\text{keep}]$ in every conditional setting? Well, I left out a tiny detail. See, the simulation is finite. There is always one biggest amount generated in the simulation. When your envelope contains that biggest amount, you should keep, according to the simulation. There may be other big amounts that are seldom generated and for which the simulated averages don't show the 11/10 ratio we expect, but the very biggest amount is the worst offender.

With an infinite simulation, all amounts would be generated an infinite number of times, so we would not get those rare-event oddities. But with a finite simulation, we cannot conclude that, given the contents of your envelope, you should always switch. There is an exception to that rule, and that exception carries a big weight in the unconditional averages (because it is a big amount).

Revisiting the Analytical Solution

Without conditioning on the contents of your envelope, can we really say that you should always switch? To reach that conclusion, we would need the unconditional $EV[\text{keep}] < EV[\text{switch}]$.

By the law of total expectation, we have

$$EV[\text{keep}] = \sum_{m=0}^{\infty} EV[\text{keep} \mid \text{your envelope} = 2^m]\, P[\text{your envelope} = 2^m]$$
$$EV[\text{switch}] = \sum_{m=0}^{\infty} EV[\text{switch} \mid \text{your envelope} = 2^m]\, P[\text{your envelope} = 2^m].$$

Each term of the $EV[\text{keep}]$ sum is strictly smaller than the corresponding term of the $EV[\text{switch}]$ sum. However, both sums diverge to infinity, so we cannot leap to the conclusion that $EV[\text{keep}] < EV[\text{switch}]$ unconditionally. The unconditional expectations $EV[\text{keep}]$ and $EV[\text{switch}]$ are in fact undefined.

I would like to point out that, as hinted by my (finite) simulation, the paradox disappears if we limit the possible number of die tosses. Let's say that if the die tossing process exceeds M tosses, then the curious man scraps the whole sequence and starts afresh.

If we condition once again on the amount you see in your envelope, we always have $EV[\text{keep}] < EV[\text{switch}]$ *except* when your envelope contains 2^M

dollars. Because of that exception, we no longer conclude that you should always switch, so there is no paradox anymore.

If we approached the two-envelope problem through the finite variation I just described, and then let $M \to \infty$, we would dodge the paradox.

Conclusion

The apparent paradox in the two-envelope problem comes from the leap from conditional expectations to unconditional expectations. The conditional expectations are finite. It makes sense to compare them. The unconditional expectations are undefined (infinite), so it does not make sense to compare them.